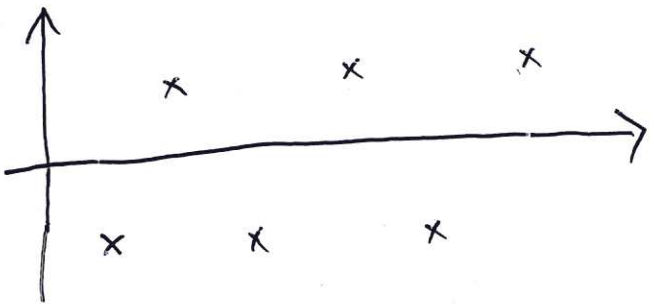
Week 2

Limit of Sequence

Last time: $\lim_{n\to\infty} \frac{1}{n} = 0$

eg $a_n = (-1)^n$



a_n does not appear to approach any real number as n approaches ∞

$\Rightarrow \lim_{n\to\infty} a_n$ does not exist (DNE)

Rmk $\lim_{n\to\infty} a_{2n} = 1$

$\lim_{n\to\infty} a_{2n-1} = -1$

eg $a_n = n^2$

$1, 4, 9, 16, 25, \dots$

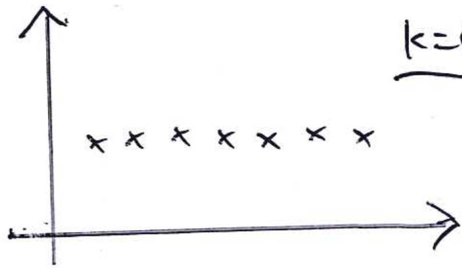
$\lim_{n\to\infty} a_n$ DNE

Rmk We can also say that $\lim_{n\to\infty} a_n = \infty$

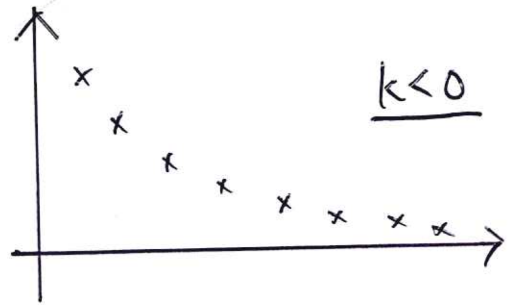
More Generally, if $a_n = n^k$

$$\lim_{n\to\infty} a_n = \begin{cases} \text{DNE } (\infty) & \text{if } k>0 \\ 1 & \text{if } k=0 \\ 0 & \text{if } k<0 \end{cases}$$

$n^k = \frac{1}{n^{-k}}$



k=0

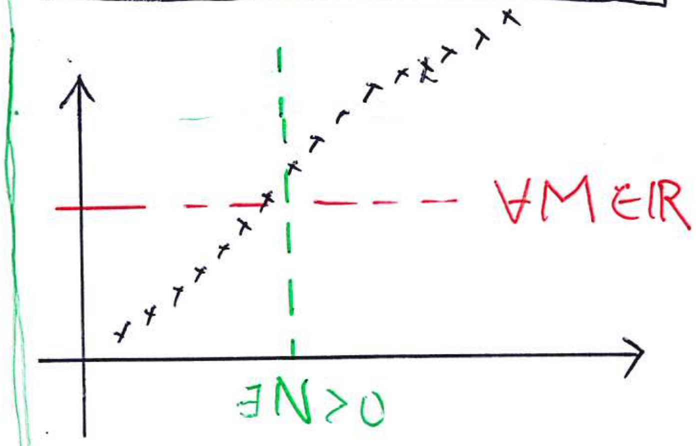


k<0

Defn (Optional)

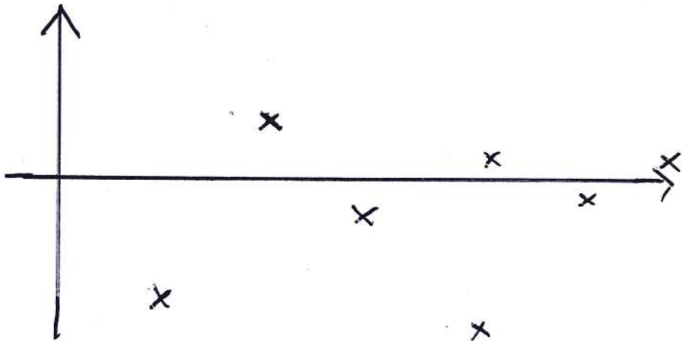
We say that $\lim_{n\to\infty} a_n = \infty$ if

$$\boxed{\forall M \in \mathbb{R}, \exists N > 0 \text{ such that if } n > N, a_n > M}$$

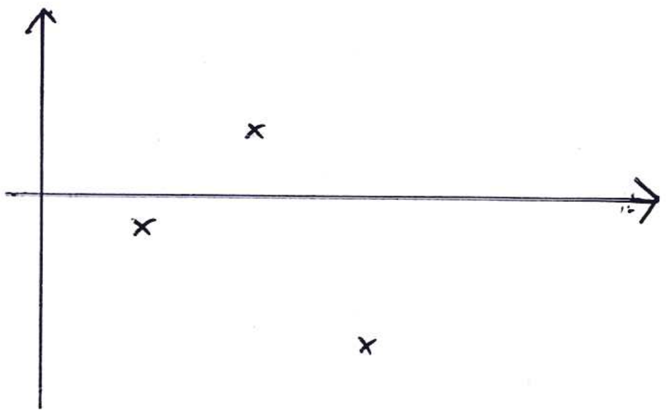


eg $a_n = b^n$, where $b \in \mathbb{R}$ is constant

$$\lim_{n\to\infty} a_n = \begin{cases} \text{DNE } (\infty) & \text{if } b>1 \\ 1 & \text{if } b=1 \\ 0 & \text{if } -1<b<1 \\ \text{DNE} & \text{if } b \le -1 \end{cases}$$



$b = -\frac{1}{2}$



$b = -2$

$+, -, \times, \div$, power of limits

Suppose $\{a_n\}$, $\{b_n\}$ are convergent. Then

1. $\lim_{n\to\infty}(a_n \pm b_n) = \lim_{n\to\infty} a_n \pm \lim_{n\to\infty} b_n$
2. $\lim_{n\to\infty}(a_n b_n) = \left(\lim_{n\to\infty} a_n\right)\left(\lim_{n\to\infty} b_n\right)$
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$ if $\lim_{n\to\infty} b_n \neq 0$
4. $\lim_{n\to\infty} a_n^k = \left(\lim_{n\to\infty} a_n\right)^k$ if $k \in \mathbb{R}$, $\lim_{n\to\infty} a_n > 0$

Rmk If $\lim = \pm\infty$,

- $\infty \pm L = \infty$
- $-\infty \pm L = -\infty$
- $\infty + \infty = \infty$
- $-\infty - \infty = -\infty$
- $L \cdot \infty = \begin{cases} \infty & \text{if } L>0 \\ -\infty & \text{if } L<0 \end{cases}$
- $\frac{L}{\pm\infty} = 0$

Indeterminant ?

$\infty - \infty$

$\frac{\pm\infty}{\pm\infty}$ $\quad \frac{0}{0}$

$0 \cdot \infty$

eg $\lim_{n\to\infty}\left(\frac{3}{n}-7+2^{-n}\right)$

$\lim_{n\to\infty}\frac{3}{n}=0 \quad \lim_{n\to\infty}7=7$

$\lim_{n\to\infty}2^{-n}=\lim_{n\to\infty}\frac{1}{2^n}=0$

$\Rightarrow \lim_{n\to\infty}\left(\frac{3}{n}-7+2^{-n}\right)$

$=0-7+0=-7$

eg Find $\left(\frac{\infty}{\infty}\right)$

i). $\lim_{n\to\infty}\frac{n+1}{3n-1}$

ii). $\lim_{n\to\infty}\frac{2n^2+5}{n^3+1}$

iii). $\lim_{n\to\infty}\frac{n^2-1}{n+3}$

Sol. i) $\lim_{n\to\infty}\frac{n+1}{3n-1}=\lim_{n\to\infty}\frac{1+\frac{1}{n}}{3-\frac{1}{n}}=\frac{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)}{\lim_{n\to\infty}\left(3-\frac{1}{n}\right)}=\frac{1}{3}$

ii) $\lim_{n\to\infty}\frac{2n^2+5}{n^3+1}=\lim_{n\to\infty}\frac{\frac{2}{n}+\frac{5}{n^3}}{1+\frac{1}{n^3}}=\frac{\lim_{n\to\infty}\left(\frac{2}{n}+\frac{5}{n^3}\right)}{\lim_{n\to\infty}\left(1+\frac{1}{n^3}\right)}$

$=\frac{\lim_{n\to\infty}\frac{2}{n}+\lim_{n\to\infty}\frac{5}{n^3}}{\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{1}{n^3}}=\frac{0+0}{1+0}=0$

iii) $\lim_{n\to\infty}\frac{n^2-1}{n+3}=\lim_{n\to\infty}\frac{n-\frac{1}{n}}{1+\frac{3}{n}}=\frac{\infty}{1}=\infty$

Rmk Different results by comparing degrees of denominator and numerator.

eg $\lim_{n\to\infty}\frac{3n}{\sqrt{4n^2+7n}}=\lim_{n\to\infty}\frac{3}{\sqrt{4+\frac{7}{n}}}=\frac{3}{\sqrt{4+0}}=\frac{3}{2}$

eg $\lim_{n\to\infty}\left(n-\sqrt{n^2+4n}\right)$

$$= \lim_{n\to\infty}\left(n-\sqrt{n^2+4n}\right)\cdot\frac{n+\sqrt{n^2+4n}}{n+\sqrt{n^2+4n}}$$

$$= \lim_{n\to\infty}\frac{n^2-(n^2+4n)}{n+\sqrt{n^2+4n}}$$

$\boxed{(a-b)(a+b) = a^2-b^2}$

$$= \lim_{n\to\infty}\frac{-4n}{n+\sqrt{n^2+4n}}$$

$$= \lim_{n\to\infty}\frac{-4}{1+\sqrt{1+\frac{4}{n}}}$$

$$= \frac{-4}{1+\sqrt{1+0}} = -2$$

Monotonic / bounded Sequence

Defn A sequence $\{a_n\}$ is said to be

(i) increasing if $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$

(ii) decreasing if $a_{n+1} \leq a_n \quad \forall n \in \mathbb{N}$

(iii) monotonic if it is increasing or decreasing

(iv) bounded above if $\exists M \in \mathbb{R}$ such that

$$a_n \leq M \quad \forall n \in \mathbb{N}$$

M is called an upper bound of $\{a_n\}$

(v) bounded below if $\exists M \in \mathbb{R}$ such that

$$a_n \geq M \quad \forall n \in \mathbb{N}$$

a lower bounded

(vi) bounded if $\exists M \in \mathbb{R}$ such that

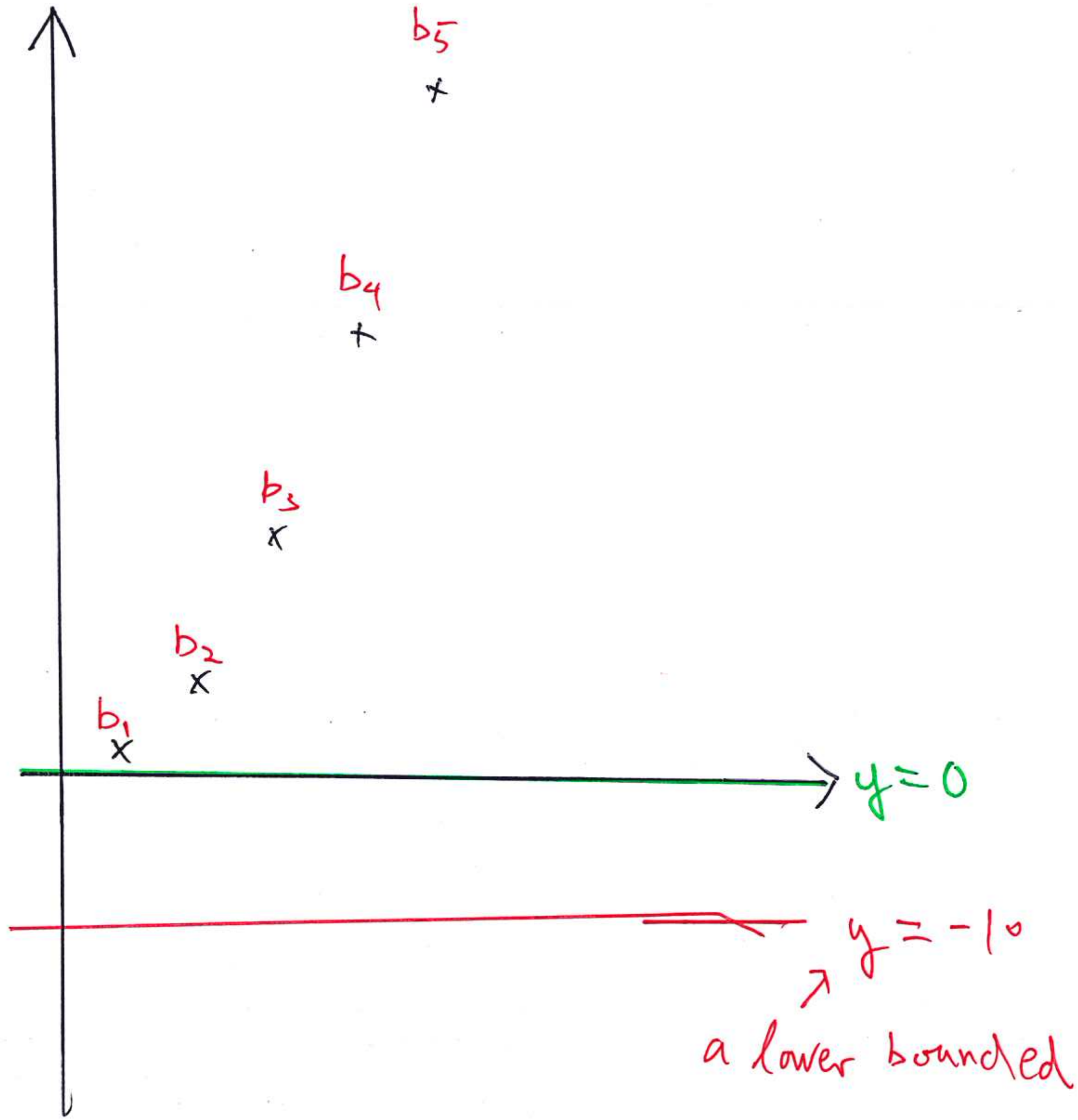
$$|a_n| \leq M \quad \forall n \in \mathbb{N}$$

Rmk

Bounded $\iff$ both bounded above and below

(vi) $\iff$ (iv) + (v)

	$a_n = \frac{1}{n}$	$b_n = n^2$
increasing	✗	✓
decreasing	✓	✗
monotonic	✓	✓
bounded above	✓	✗
bounded below	✓	✓
bounded	✓	✗



Both 0 and -10 are lower bound for b_n

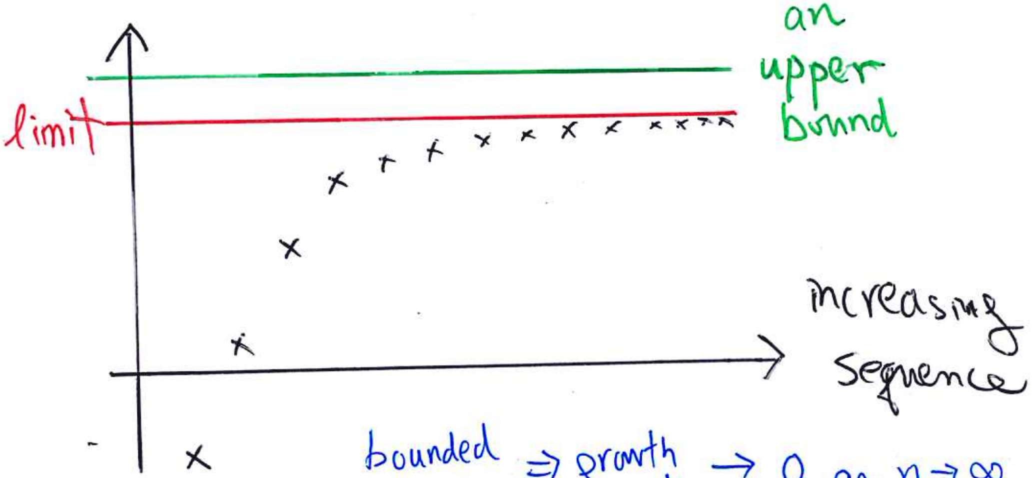
Thm (Monotone Convergence theorem)

If a sequence $\{a_n\}$ is either

$$\begin{cases} \text{bounded above and increasing} \\ \text{bounded below and decreasing} \end{cases}$$

then $\lim_{n\to\infty} a_n$ exists

Rmk The theorem doesn't tell us what the limit is



bounded above $\Rightarrow$ growth rate $\to 0$ as $n \to \infty$

eg. 1 Let $a_1 = 1$, $a_n = \sqrt{1 + a_{n-1}}$ for $n \geq 2$

Show that it is convergent and find its limit.

Sol $a_1 = 1$, $a_2 = \sqrt{2}$, $a_3 = \sqrt{1+\sqrt{2}}$ seems $\nearrow$

Step 1 Show that a_n is increasing by induction

ie. Show that $a_{n+1} \geq a_n$ for all n

(a) For $n=1$, $a_2 = \sqrt{2} \geq 1 = a_1$

(b) Assume that $a_{k+1} \geq a_k$, then

$$a_{k+2} = \sqrt{1 + a_{k+1}} \quad (\because \text{By definition})$$

$$\geq \sqrt{1 + a_k} \quad (\text{Induction assumption})$$

$$= a_{k+1}$$

Induction $\Rightarrow a_{n+1} \geq a_n$ for all n

Step 2 Show that $a_n \leq 2$ by induction

(a) For $n=1$, $a_1 = 1 \leq 2$

(b) Assume that $a_k \leq 2$, then

$$a_{k+1} = \sqrt{1+a_k} \quad \text{(by definition)}$$
$$\leq \sqrt{1+2} \quad \text{(Induction assumption)}$$
$$\leq 2$$

Induction $\Rightarrow$ $a_n \leq 2$ for all n

a_n is both increasing and bounded above

$$\boxed{\text{Monotone Convergence Thm} \Rightarrow \lim_{n\to\infty} a_n \text{ exists}}$$

Step 3 Let $\lim_{n\to\infty} a_n = L$

$$a_n = \sqrt{1+a_{n-1}}$$

Take $\lim_{n\to\infty}$ on both sides

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt{1+a_{n-1}}$$
$$L = \sqrt{1+\lim_{n\to\infty} a_{n-1}}$$
$$= \sqrt{1+L}$$

$\Rightarrow L^2 = 1+L$

$\Rightarrow L^2 - L - 1 = 0$

$\Rightarrow \left(L-\frac{1}{2}\right)^2 - \frac{5}{4} = 0$

$\Rightarrow L = \frac{1}{2} \pm \frac{\sqrt{5}}{2}$

$\because a_n > 0 \ \forall n$

$\therefore L \geq 0$

$\frac{1}{2} - \frac{\sqrt{5}}{2}$ is rejected

$\Rightarrow \boxed{L = \frac{1}{2} + \frac{\sqrt{5}}{2}}$

eg

$$a_1 = 1, \quad a_{n+1} = \frac{3a_n + 1}{a_n + 1} \quad \text{for } n \geq 1$$

Show that $\lim_{n\to\infty} a_n$ exists and find its value.

Sol The first few terms:

$$1, 2, \frac{7}{3}, \frac{12}{5}, \ldots$$

Step 1 Show that $\{a_n\}$ is bounded above

$$a_{n+1} = \frac{3a_n + 3 - 2}{a_n + 1} = 3 - \frac{2}{a_n + 1}$$

Clearly, $a_n \geq 0 \Rightarrow a_{n+1} \leq 3$ for $n \geq 1$

Also $a_1 = 1 \leq 3 \Rightarrow \boxed{a_n \leq 3 \quad \forall n}$

Step 2 Show that a_n is increasing

Prove $a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$ by induction!

For $n=1$, $a_2 = 2 \geq 1 = a_1 \Rightarrow$ true for $n=1$

Assume that $a_{k+1} \geq a_k$, then

$$a_{k+2} - a_{k+1} = \frac{3a_{k+1} + 1}{a_{k+1} + 1} - \frac{3a_k + 1}{a_k + 1}$$

$$= \frac{(3a_{k+1} + 1)(a_k + 1) - (3a_k + 1)(a_{k+1} + 1)}{(a_{k+1} + 1)(a_k + 1)}$$

$$= \frac{2(a_{k+1} - a_k)}{(a_{k+1} + 1)(a_k + 1)} \geq 0 \Rightarrow a_{k+2} \geq a_{k+1}$$

Induction $\Rightarrow a_{n+1} \geq a_n \quad \forall n \in \mathbb{N}$, ie $\{a_n\}$ is increasing

Step 3 $\{a_n\}$ increasing and bounded above

Monotone Convergence thm $\Rightarrow \{a_n\}$ is convergent. Let $L = \lim_{n\to\infty} a_n$.

Then $a_{n+1} = \frac{3a_n + 1}{a_n + 1} \Rightarrow \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{3a_n + 1}{a_n + 1}$

$$\Rightarrow L = \frac{3L + 1}{L + 1} \Rightarrow L^2 - 2L - 1 = 0 \Rightarrow \boxed{L = 1 + \sqrt{2}} \; (\because L \geq 0)$$

The number e

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \approx 2.7182818\ldots\ldots$$

How to prove the limit exist?

Ans By Monotone Convergence thm.

Let $a_n = \left(1+\frac{1}{n}\right)^n$. Then

$$a_n = \sum_{k=0}^{n} C^n_k \left(\frac{1}{n}\right)^k \qquad \left(C^n_k = \frac{n!}{k!(n-k)!} \right)$$

$$= 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\left(\frac{1}{n}\right)^2 + \frac{n(n-1)(n-2)}{3!}\left(\frac{1}{n}\right)^3 + \cdots + \frac{n(n-1)\cdots(n-(n-1))}{n!}\left(\frac{1}{n}\right)^n$$

$$= \underbrace{1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)}_{n+1 \text{ terms}} \quad (*)$$

Similarly,

$$a_{n+1} = \underbrace{1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \cdots + \frac{1}{n!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n-1}{n+1}\right) + \frac{1}{(n+1)!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{n}{n+1}\right)}_{n+2 \text{ terms}}$$

Comparing first $n+1$ terms $\Rightarrow$ $\boxed{a_{n+1} \geq a_n}$

Also $(*)$ $\Rightarrow$

$$a_n \leq 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

$$\leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}}$$

$$\leq 1 + 2 = 3$$

$\therefore \boxed{a_n \leq 3}$ $\overset{\text{MCT}}{\Rightarrow} \lim_{n\to\infty} a_n$ exists

Thm (Sandwich theorem)

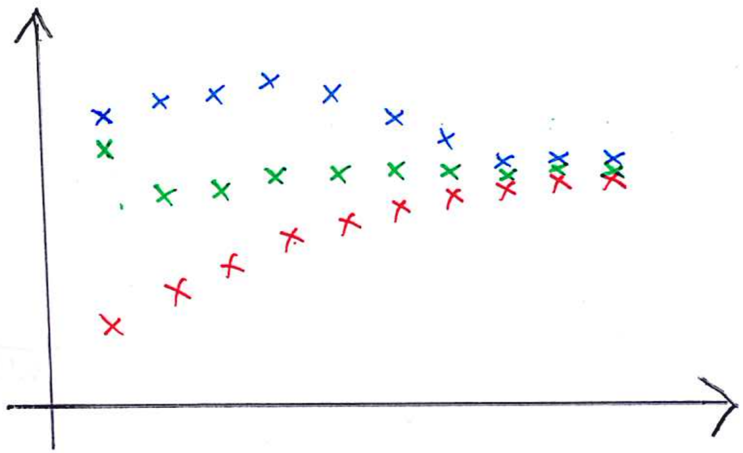
Let $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ be sequences.

Suppose that

$$a_n \le b_n \le c_n \quad \forall n \in \mathbb{N}$$

and $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} c_n = L$

Then $\lim\limits_{n\to\infty} b_n = L$



eg Show that the following limits exist

① $\lim\limits_{n\to\infty} \dfrac{n+(-1)^n}{n+1}$ ② $\lim\limits_{n\to\infty} \dfrac{\sin(n^2)}{2n+1}$

Sol ①

$$\frac{n-1}{n+1} \le \frac{n+(-1)^n}{n+1} \le \frac{n+1}{n+1} = 1$$

$$\lim_{n\to\infty} \frac{n-1}{n+1} = \lim_{n\to\infty} \frac{1-\frac{1}{n}}{1+\frac{1}{n}} = \frac{1-0}{1+0} = 1 = \lim_{n\to\infty} 1$$

Sandwich theorem $\Rightarrow \lim\limits_{n\to\infty} \dfrac{n+(-1)^n}{n+1} = 1$

② $-1 \le \sin(n^2) \le 1$, $2n+1 > 0$

$$\Rightarrow \frac{-1}{2n+1} \le \frac{\sin(n^2)}{2n+1} \le \frac{1}{2n+1}$$

Also, $\lim\limits_{n\to\infty} \dfrac{-1}{2n+1} = \lim\limits_{n\to\infty} \dfrac{1}{2n+1} = 0$

Sandwich theorem $\Rightarrow \lim\limits_{n\to\infty} \dfrac{\sin(n^2)}{2n+1} = 0$

Corollary (*) Let $\{a_n\}$ be a sequence

Then

$$\lim_{n\to\infty} a_n = 0 \iff \lim_{n\to\infty} |a_n| = 0$$

Pf

($\Rightarrow$) Suppose $\lim_{n\to\infty} a_n = 0$

then $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \sqrt{a_n^2}$

$$= \sqrt{\left(\lim_{n\to\infty} a_n\right)^2}$$

$$= \sqrt{0^2} = 0$$

($\Leftarrow$) Suppose $\lim_{n\to\infty} |a_n| = 0$.

$$-|a_n| \le a_n \le |a_n|$$

$$\lim_{n\to\infty} -|a_n| = -\lim_{n\to\infty} |a_n| = -0 = 0$$

$$\lim_{n\to\infty} |a_n| = 0$$

Sandwich thm $\Rightarrow \lim_{n\to\infty} a_n = 0$

Corollary Suppose that

$\{a_n\}$ is bounded, $\lim_{n\to\infty} b_n = 0$

Then $\lim_{n\to\infty} a_n b_n = 0$

Pf $\{a_n\}$ is bounded

$\Rightarrow \exists M \in \mathbb{R}$ such that $|a_n| < M \quad \forall n \in \mathbb{N}$

$\Rightarrow 0 \le |a_n b_n| = |a_n||b_n| \le M|b_n|$

$\lim_{n\to\infty} 0 = 0$, $\lim_{n\to\infty} M|b_n| = M \lim_{n\to\infty} |b_n| = 0$

by (*)

$\Rightarrow \lim_{n\to\infty} |a_n b_n| = 0$

$\Rightarrow \lim_{n\to\infty} a_n b_n = 0$

by (*)